JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 125, No. 3, pp. 673–695, June 2005 (© 2005)
DOI: 10.1007/s10957-005-2095-1

Location Problems with Different Norms for Different Points[1,2]

P. L. PAPINI[3] AND J. PUERTO[4]

Communicated by R. Conti

Abstract. Given a finite set $A = \{a_1, \ldots, a_n\}$ in a linear space X, we consider two problems. The first problem consists of finding the points minimizing the maximum distance to the points in A; the second problem looks for the points that minimize the average distance to the points in A. In both cases, we assume that the distances at different points are defined as

$$d(x, a_i) = \|x - a_i\|_i, \quad \text{for } i = 1, \ldots, n,$$

with norms $\|\cdot\|_i$ defined on X. The use of different norms to measure distances from different points allows us to extend some results that hold in the single-norm case, while some strange and rather unexpected facts arise in the general case.

Key Words. Location problems, centers, medians, different norms.

1. Introduction

Let X be a vector space over the real field $\mathbb{R}$ and let

$$A = \{a_1, \ldots, a_n\} \subset X, \quad n \geq 2, \quad a_i \neq a_j \quad \text{for } i \neq j,$$

be a finite subset of X. We associate a norm $\|\cdot\|_i$ with each point $a_i, i = 1, \ldots, n$. Moreover, we assume that the normed space $X_i = (X, \|\cdot\|_i)$ is

[1]The research of the first author was supported by the Italian National Group GNAMPA.
[2]The research of the second author was supported by the Spanish Ministry of Science and Technology through Grant BFM2001-2378, MTM2004:0909.
[3]Professor, Department of Mathematics, University of Bologna, Bologna, Italy.
[4]Professor, Faculty of Mathematics, University of Seville, Seville, Spain.

complete for every i. We denote by $B_i(x,r)$ the ball of radius r centered at x,

$$B_i(x,r)=\{y\in X:\|x-y\|_i\leq r\}.$$

To simplify notation, we refer to the unit ball centered at zero as B_i. Each norm $\|\cdot\|_i$ has associated the so-called dual norm $\|\cdot\|_i^o$ defined by its unit dual ball

$$B_i^o=\{u\in X^*:u(x)\leq 1,\forall x\in B_i\}.$$

Also, we denote by $\bar{C}$ and $\text{int}(C)$ the topological closure and the interior of the set C, respectively.

Consider on X a convex function,

$$x\to f(A,x)=f(\|a_1-x\|_1,\dots,\|a_n-x\|_n).$$

In this paper, we investigate the problem $\inf_{x\in X} f(A,x)$ by considering the two most common functions in this context:

$$f_1(A,x):x\to\sum_{i=1}^{n}\|a_i-x\|_i, \tag{1}$$

$$f_2(A,x):x\to\max_{1\leq i\leq n}\|a_i-x\|_i. \tag{2}$$

Several natural situations fall into this formulation; see Ref. 1 or Ref. 2 for the study of this type of problems in location theory. The goal of this paper is to show how some known results concerning problems (1) and (2), when only one norm is used, can be extended to this more general situation, while at the same time some strange and rather unexpected facts arise in the general case. We note in passing that even the use of (different) Hilbertian norms not always helps. Recall that a norm $\|\cdot\|_i$ is Hilbertian if it is induced by an inner product, that we denote by $(\cdot,\cdot)_i$.

Apparently, until now little attention has been devoted to these problems. Nevertheless, one can find some papers that deal with different norms such as Refs. 1–11. Reference 1 deals with algorithmic approaches to some of these problems in the plane; Refs. 2 and 3, where more generally gauges are used, describe the set of optimal points to some of these location problems; Ref. 4 deals with Pareto optima and the median problem (also this one considers gauges); Ref. 5 describes the properties of the optimal solutions for both medians and centers; Ref. 6 considers also different gauges (the interested reader may see also the references therein); Refs. 7–10 cover research based on applications of general convex and global optimization techniques to location analysis; Ref. 11 deals with sensitivity analysis with respect to p in Weber problems with different l_p norms. For the sake of completeness, we recall that attempts considering Pareto solutions with different norms have been done also in Refs. 12–15.

Minimizing $f_j(A,\cdot)$, $j=1,2$, is equivalent to the following problem: given the space $Y=(X_1\times\cdots\times X_n)$ endowed with the norm

$$|||y|||_1=\sum_{i=1}^{n}\|x_i\|_i,\quad \text{for } j=1,$$

or

$$|||y|||_2=\max_{1\le i\le n}\|x_i\|_i,\quad \text{for } j=2,$$

and given

$$a=(a_1,\ldots,a_n)\in X_1\times\cdots\times X_n,$$

look for the best approximation to a from the diagonal of Y. In other words, look for a point $\mu=(m,\ldots,m)$ [respectively, $\gamma=(c,\ldots,c)$] in Y realizing the distance between a and the diagonal.

Note that minimizing f_1 or f_2 is a convex problem; therefore, for any $\varepsilon>0$ and $j=1,2$, the sets

$$M_j^{\varepsilon}(A)=\{y\in X: f_j(y)\le \inf_{x\in X} f_j(A,x)+\varepsilon\}$$

are closed and convex. In addition, by the definition of the functions f_1 and f_2, these sets are also bounded in Y. Therefore, the possibly empty set of minimizers

$$M_j(A)=\{y\in X: f_j(A,y)=\inf_{x\in X} f_j(A,x)\}$$

is closed, bounded, and convex; see e.g. Ref. 16 for examples of empty sets of solutions.

If all the spaces

$$X_i=(X,\|\cdot\|_i),\quad \text{for } i=1,\ldots,n,$$

are reflexive, then the space Y endowed with both norms is reflexive and $M_j(A)\neq\emptyset$, for $j=1,2$ and any finite set $A\subset X$. If all the norms $\|\cdot\|_i$ are strictly convex, then at most one minimum exists for $f_j(A,x)$, $j=1,2$, unless A is contained in a line and $j=1$.

Apart from the above general remarks that apply both to problems (1) and (2), in the next two sections we analyze separately each of these problems.

2. Centers and Diameters

Given $x \in X$ and $A = \{a_1, \ldots, a_n\}$, let us consider

$$r(A, x) = \max_{1 \le i \le n} \|x - a_i\|_i, \qquad \text{radius of } x \text{ with respect to } A, \tag{3}$$

$$r(A) = \inf_{x \in X} r(A, X), \qquad \text{radius of } A, \tag{4}$$

$$\delta(A) = \max_{1 \le i, j \le n} \|a_i - a_j\|_i, \qquad \text{diameter of } A. \tag{5}$$

We say that $c \in X$ is a general center (or simply a center, when no confusion is possible) of A if

$$r(A, c) = r(A). \tag{6}$$

The reader may notice that this notion of center differs from that of the classical Chebyshev center and other definitions given in the literature; see Refs. 17–18.

Also, for $\varepsilon > 0$, we set

$$c^{\varepsilon}(A) = \{x \in X : r(A, x) \le r(A) + \varepsilon\}, \tag{7}$$

$$c(A) = \{x \in X : r(A, x) = r(A)\} = \bigcap_{\varepsilon > 0} c^{\varepsilon}(A). \tag{8}$$

As noticed in the introduction, $c(A)$ [also denoted as $M_2(A)$] is a possibly empty closed, bounded, convex subset of X.

For $A = \{a_1, \ldots, a_n\}$, we consider also, for i fixed between 1 and n, the numbers $r_i(A, x), r_i(A)$, and $\delta_i(A)$ when only the norm determined by i is considered. Also, c_i is an i-center of A if

$$r_i(A, c_i) = r_i(A).$$

Here, c_i is the usual center of A in the space X_i.

Clearly, for every finite set A, we have

$$\delta_i(A)/2 \le r_i(A) \le r_i(A, a_i) \le \delta_i(A) \le \delta(A);$$

also,

$$r(A) \le r(A, x_i) \le \delta(A) = \max_{1 \le i \le n} \delta_i(A) \le 2 \max_{1 \le i \le n} r_i(A).$$

Nevertheless, despite the above inequalities, it is not easy to estimate the value $r(A)$. The following example proves that

$$r(A) \not\le \max_{1 \le i \le n} r_i(A).$$

Example 2.1. Consider in $\mathbb{R}^6$ the vectors

$$a_1=(1,0,0,0,0,0),\quad a_2=(0,1,0,0,0,0),\quad a_3=(0,0,1,0,0,0),$$
$$a_4=(0,0,0,5,0,0),\quad a_5=(0,0,0,0,5,0),\quad a_6=(0,0,0,0,0,5),$$

with

$$\|x\|_1=\|x\|_2=\|x\|_3=\sum_{i=1}^{6}|x_i|,$$
$$\|x\|_4=\|x\|_5=\|x\|_6=(8/5)\max_{1\le i\le 6}|x_i|.$$

Take

$$c=(0,0,0,0,0,0)$$

to check that

$$r_1(A)=r_2(A)=r_3(A)\le 5.$$

In addition, take

$$c=(0,0,0,5/2,5/2,5/2)$$

to check that

$$r_4(A)=r_5(A)=r_6(A)\le 4.$$

Let us denote by $\bar{c}$ a general center. In order to have $r(A)\le 5$, we must take for the last three components of $\bar{c}$ a value α so that

$$(5-\alpha)8/5=8-1.6\alpha\le 5,$$

that is, $\alpha\ge 15/8$. But now, if we consider a_1, we get

$$\|a_1-\bar{c}\|_1\ge 3\times(15/8)>5,$$

so $r(A)>5$.

The next result extends Theorem 1 in Ref. 19; generalizations (in case of a single norm) were indicated in Section 4 of Ref. 20.

Theorem 2.1. Let c be a center of $A=\{a_1,\dots,a_n\}$ and let $A_c=\{a_i\in A:\|c-a_i\|_i=r(A)\}$. Then, we have $r(A)=r(A_c)$. Hence, c is a center also for A_c.

Proof. Clearly, since $A_c \subseteq A$, we have

$$r(A_c) \leq r(A).$$

Assume that

$$r(A_c) = r(A) - \varepsilon, \quad \text{for some } \varepsilon > 0.$$

Let

$$\sigma = \inf\{r(A) - \|c - a_i\|_i : a_i \in A \backslash A_c\}.$$

Take $c' \in X$ such that

$$r(A_c, c') \leq r(A) - \varepsilon/2.$$

Now, take $\lambda_i \in (0, 1)$ such that

$$\lambda_i \|c - c'\|_i < \sigma, \quad i = 1, \ldots, n;$$

then, let

$$\lambda = \min_{1 \leq i \leq n} \{\lambda_i\} \quad \text{and} \quad c'' = c + \lambda(c' - c).$$

For $a_i \in A_c$, we have

$$\begin{aligned} \|c'' - a_i\|_i &= \|(1 - \lambda)c + \lambda c' - a_i\|_i \\ &\leq (1 - \lambda)\|c - a_i\|_i + \lambda\|c' - a_i\|_i \\ &\leq (1 - \lambda)r(A) + \lambda(r(A) - \varepsilon/2) \\ &< r(A); \end{aligned}$$

if $a_i \in A \backslash A_c$, we have

$$\begin{aligned} \|c'' - a_i\|_i &\leq \|c - a_i + \lambda(c' - c)\|_i \\ &< r(A) - \sigma + \sigma \\ &= r(A). \end{aligned}$$

Thus,

$$\|c'' - a_i\|_i < r(A), \quad \text{for any } i.$$

This contradiction proves the thesis. □

Corollary 2.1. If c is a center of $A = \{a_1, \ldots, a_n\}$, then $\|c - a_i\|_i = r(A)$ for at least two indexes.

Remark 2.1. If a single norm is used and X is strictly convex, then there is at most one center c of A such that either

$$\|c-a_{i_j}\|=r(A),$$

for a pair a_{i_1}, a_{i_2} determining a diameter of $A[c=(a_{i_1}+a_{i_2})/2]$, or there are at least three points $a_{i_1}, a_{i_2}, a_{i_3}$ such that

$$\|c-a_{i_1}\|=\|c-a_{i_2}\|=\|c-a_{i_3}\|=r(A)$$

and

$$c\in \mathrm{co}(\{a_{i_1}, a_{i_2}, a_{i_3}\}).$$

The reader can see in Example 2.4 that this is not true when different norms are used. See also Figure 2.

Not always centers of sets exist. Following the lines of the proof given in Ref. 20 for usual centers (with respect to a given norm), we can give the following more general result.

Proposition 2.1. Given $A=\{a_1,\ldots,a_n\}$, let $r(A)=r$. Then, for every $\sigma\in(0,r]$, there is a subset A^σ of A such that $r(A^\sigma)\in[r-\sigma,r]$ and, for some $c^\sigma\in X, r-\sigma\le\|c^\sigma-a_i\|_i\le r+\sigma$ for all $a_i\in A^\sigma$.

Chebyshev centers in Hilbert spaces can be characterized in the following way (see Theorem 2.1 in Ref. 21).

Proposition 2.2. Let $A=\{a_1,\ldots,a_n\}$ be contained in a Hilbert space X; then, c is the (unique) center of A if and only if, for every $x\in X$, we have

$$r^2(A)+\|x-c\|^2\le r^2(A,x).$$

The extension of this result for centers in our sense (when all the norms used are Hilbertian but different) is not possible, as the following example shows.

Example 2.2. In the plane let

$$A=\{a_1,a_2,a_3\},$$

with

$$a_1=(0,0),\quad a_2=(0,3),\quad a_3=(3,0).$$

Consider the following norms:

$$\|x\|_1 = (1/2)\sqrt{x_1^2 + x_2^2}, \quad \|x\|_2 = \sqrt{x_1^2 + x_2^2}, \quad \|x\|_3 = (1/\sqrt{13})\sqrt{x_1^2 + x_2^2}.$$

The point $c = (0, 2)$ is the unique center of A and

$$r(A) = 1 = \|c - a_i\|_i, \quad i = 1, 2, 3.$$

If we take

$$x = (4/5, 13/5),$$

we have

$$\|x - a_i\|_i^2 \leq \|x - a_1\|_1^2 = (1.3)^2 + (0.4)^2 = 1.85;$$

also,

$$\|x - c\|_2^2 = 1.$$

Therefore,

$$r^2(A) + \|x - c\|_2^2 > [\max_{1 \leq i \leq 3} \|x - a_i\|_i]^2.$$

The catalogue of the unexpected facts can be enlarged from other well-known results of the single-norm case. Indeed, for a single norm, we have always

$$\delta(A) \leq 2r(A).$$

Nevertheless, apart from the direct inequalities

$$\delta_i(A) \leq 2r_i(A),$$

nothing similar can be expected in general as shown by Example 2.3. Moreover, it is not difficult to see that the equidistant set,

$$E(a_i, a_j) = \{x \in X : \|x - a_i\|_i = \|x - a_j\|_j\},$$

even if $\|\cdot\|_i, \|\cdot\|_j$ are Euclidean, is not in general a straight line.

Example 2.3. Let X be a plane; let

$$A = \{a_1, a_2, a_3\},$$

with

$$a_1 = (0, 1), \quad a_2 = (1, 0), \quad a_3 = (0, 0).$$

Let

$$\|x\|_1 = \|x\|_2 = \sqrt{x_1^2 + x_2^2}, \quad \|x\|_3 = n\sqrt{x_1^2 + x_2^2},$$

so that

$$\|a_1 - a_2\|_1 = \|a_2 - a_1\|_2 = \sqrt{2},$$
$$\|a_1 - a_3\|_1 = \|a_2 - a_3\|_2 = 1,$$
$$\|a_3 - a_1\|_3 = \|a_3 - a_2\|_3 = n.$$

We have

$$r(A) \leq r(A, a_3) = \max(\|a_3 - a_1\|_1, \|a_3 - a_2\|_2) = 1;$$

in fact, the center is

$$[1/(1 + \sqrt{2n^2 - 1}), 1/(1 + \sqrt{2n^2 - 1})].$$

Also,

$$r(A) = n\sqrt{2}/(1 + \sqrt{2n^2 - 1}), \quad \delta(A) = n,$$

so that, for every $n \geq 1$, we get

$$\delta(A) > nr(A).$$

In addition this example shows that

$$r_3(A)\sqrt{2} = \delta(A);$$

so, for n large,

$$r_3(A) > r(A) + r_1(A) + r_2(A).$$

Remark 2.2. It is known that, if $\dim(X)=2$ or if X is Hilbertian, then there is a Chebyshev center of A contained in the convex hull of A; moreover, if A is contained in a line, then

$$r(A)=\delta(A)/2$$

and the middle point of A is a center. All these facts are not true for the centers that we are considering, as it is proved by the following example.

Example 2.4. In $X=\mathbb{R}^2$, take the points

$$a_1=(-1,1), \quad a_2=(1,-1),$$

with unit balls $B_i, i=1,2$, respectively. The unit balls translated into the corresponding points are (see Fig. 1)

$$a_1+B_1=\{(x,y)\in\mathbb{R}^2: 4(x+1)^2/(2+\sqrt{3})^2+4(y-1)^2\leq 1\},$$
$$a_2+B_2=\{(x,y)\in\mathbb{R}^2: 4(y+1)^2/(2+\sqrt{3})^2+4(x-1)^2\leq 1\}.$$

For the sake of simplicity, set

$$\alpha=2\cos((5/12)\pi)/(\sqrt{6}+\sqrt{2}).$$

In this case, the center is

$$c=(1-\alpha,1-\alpha)\cong(0.865,0.865).$$

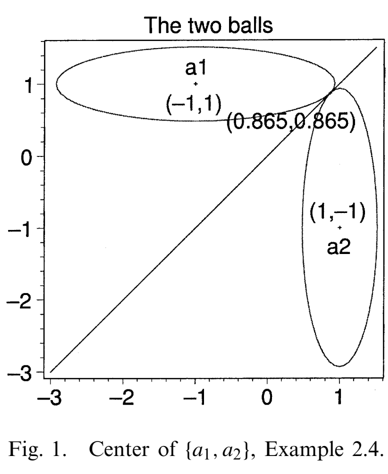

Fig. 1. Center of $\{a_1,a_2\}$, Example 2.4.

Thus, we have

$$\|a_1 - c\|_1 = \|a_2 - c\|_2 = 2\sqrt{(2-\alpha)^2/(2+\sqrt{3})^2 + \alpha^2} \cong 1.035,$$
$$r_1(A) = r_2(A) = 2\sqrt{(2+\sqrt{3})^{-2} + 1} \cong 2.071.$$

Hence, we have

$$r(A) = 1 < \min\{r_1(A), r_2(A)\}.$$

In addition , we take

$$a_3 = (-1, -1),$$

with

$$a_3 + B_3 = \left\{ (x, y) \in \mathbb{R}^2 : \rho^2 (x+1, y+1) \begin{bmatrix} 1/16 + \sqrt{2}/4 & -1/16 + \sqrt{2}/4 \\ -1/16 + \sqrt{2}/4 & 1/16 + \sqrt{2}/4 \end{bmatrix} \begin{pmatrix} x+1 \\ y+1 \end{pmatrix} \leq 1 \right\},$$

where

$$\rho = \left[2\sqrt{(2-\alpha)^2/(2+\sqrt{3})^2 + \alpha^2} \right] / \alpha \sqrt[4]{2}.$$

We have

$$\|a_3 - c\|_3 = 2\sqrt{(2-\alpha)^2/(2+\sqrt{3})^2 + \alpha^2} \cong 1.035.$$

Therefore, c is also the center of $\{a_1, a_2, a_3\}$. Nevertheless, $c \notin \mathrm{co}(\{a_1, a_2, a_3\})$; see Figure 2.

3. Medians

Given $x \in X$ and $A = \{a_1, \ldots, a_n\}$, let us consider

$$\mu(A, x) = (1/n) \sum_{1 \leq i \leq n} \|x - a_i\|_i, \text{ average distance of } x \text{ with respect to } A, \quad (9)$$
$$\mu(A) = \inf_{x \in X} \mu(A, x). \quad (10)$$

We say that $m \in X$ is a general median (a median, when no confusion is possible) of A if

$$\mu(A, m) = \mu(A). \quad (11)$$

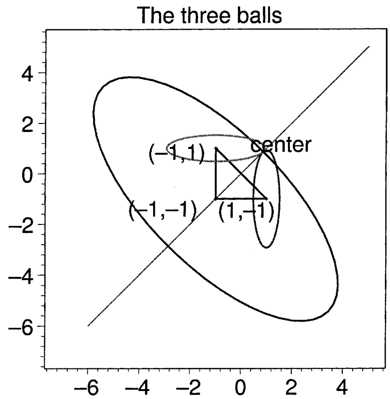

Fig. 2. Center of $\{a_1, a_2, a_3\}$ does not belong to $\mathrm{co}(\{a_1, a_2, a_3\})$, Example 2.4.

A median will be called also a minisum point. The (possible empty) set of general medians is denoted by

$$M(A) = \{x \in X : \mu(A) = \mu(A, x)\}. \tag{12}$$

By examining some of the properties that have standard Chebyshev centers and medians (see e.g. Ref. 22), we realize that most of them do not extend to general centers and medians. This fact gives rise to counterintuitive situations. For example, Example 2.4 shows that

$$(\|y-a\|_1 = \|y-a\|_2,\ y = (a_1+a_2)/2) \not\Rightarrow (\text{y is a median or a center}).$$

The reader may notice that one has the same conclusion, even if $\dim(X) = 2$ and the norms used are Hilbertian. The same example shows as well that

$$(y \text{ is a median and } \|y-a_1\|_1 = \|y-a_2\|_2) \not\Rightarrow y = (a_1+a_2)/2.$$

Nevertheless, there are also some positive results. We start with a simple inequality.

Proposition 3.1. If $x, y \in X$, then $(1/n)\sum_{i=1}^{n} \|x-y\|_i \leq \mu(A, x) + \mu(A, y)$.

Proof. We have that

$$\|x-y\|_i \leq \|x-a_i\|_i + \|a_i - y\|_i, \quad \text{for every } i = 1, \ldots, n.$$

By adding on i and dividing by n, we obtain the inequality. □

Corollary 3.1. If m_1, m_2 are medians of A, then

$$(1/n)\sum_{i=1}^{n}\|m_1-m_2\|_i \leq 2\mu(A).$$

Remark 3.1. The inequalities obtained in Proposition 3.1 and in its corollary are sharp, also when a single norm is used. However, we cannot expect in general that

$$\|m_1-m_2\|_i \leq 2\mu(A), \quad \text{for every } i,$$

as the following example shows.

Example 3.1. Take

$$A=\{a_1, a_2, a_3, a_4\},$$

where

$$a_1=(1,0), \quad a_2=(-1,0), \quad a_3=(3/2,0), \quad a_4=(-3/2,0),$$

X being a plane and

$$\|x\|_1=\|x\|_2=\max_{i=1,2}|x_i|, \quad \|x\|_3=\|x\|_4=(1/2)\max_{i=1,2}|x_i|.$$

If

$$m_1=(0,1), \quad m_2=(0,-1),$$

then we have

$$\mu(A,m_1)=\mu(A,m_2)=(1/4)(1+1+3/4+3/4)=7/8.$$

Hence,

$$\|m_1-m_2\|_i=2, \quad \text{for } i=1,2.$$

As in the case of general centers, there are not easy estimates of the value $\mu(A)$. The following example proves that

$$\mu(A) \not\leq \max_{i\leq i\leq n}\mu_i(A).$$

Example 3.2. Continuation of Example 2.1. Take

$$m=(0,0,0,0,0,0), \quad \text{for } i=1,2,3,$$

then,

$$\|m-a_1\|_i=\|m-a_2\|_i=\|m-a_3\|_i=1,$$
$$\|m-a_4\|_i=\|m-a_5\|_i=\|m-a_6\|_i=5.$$

Therefore

$$\mu_i(A)=\mu_i(A,m)=3.$$

For $i=4,5,6$, take

$$m'=(0,0,0,5/2,5/2,5/2),$$

then,

$$\|m'-a_1\|_i=\|m'-a_2\|_i=\|m'-a_3\|_i=\|m'-a_4\|_i=\|m'-a_5\|_i=\|m'-a_6\|_i=4.$$

Thus,

$$\mu_i(A,m')=4, \quad i=4,5,6.$$

In fact, $\mu_i(A)=4$ and m' is a median.

On the other hand,

$$m''=(0,0,0,0,0,0)$$

realizes the general median of A and

$$\|m''-a_1\|_1=\|m''-a_2\|_2=\|m''-a_3\|_3=1,$$
$$\|m''-a_4\|_4=\|m''-a_5\|_5=\|m''-a_6\|_6=8.$$

Hence,

$$\mu(A)=\mu(A,m'')=4.5>\max_{i=1,\ldots,6}\mu_i(A).$$

Other properties of the set of general medians are the following.

Proposition 3.2.

(i) If $m\in M(A)$ and $\|m-a_i\|_i=\text{const}$, for all $i=1,\ldots,n$, then $m\in c(A)$.

(ii) If $m \in M(A)$, then $m \in M(A \cup \{m\})$ for any norm $\|\cdot\|_0$ associated with m.
(iii) If $m \in M(A)$ and y_i is a point in the open segment between m and a_i, $i = 1, \ldots, n$, that has associated the same norm as a_i, then $m \in M(\{m, y_1, \ldots, y_n\})$ for any norm $\|\cdot\|_0$ associated with m.

Proof.

(i) It is trivial.
(ii) For any $x \in X$, take $a_0 = m$ and then

$$\sum_{i=0}^{n} \|m - a_i\|_i = \sum_{i=1}^{n} \|m - a_i\|_i = \min_{x \in X} \sum_{i=1}^{n} \|x - a_i\|_i \leq \sum_{i=0}^{n} \|x - a_i\|_i.$$

(iii) Set

$$y_i = m + \lambda_i (a_i - m), \quad 0 < \lambda_i < 1.$$

We have

$$\sum_{i=1}^{n} \|m - y_i\|_i = \sum_{i=1}^{n} (\|a_i - m\|_i - \|a_i - y_i\|_i) \leq \sum_{i=1}^{n} (\|a_i - x\|_i - \|a_i - y_i\|_i) \leq \sum_{i=1}^{n} \|x - y_i\|_i, \forall x.$$

Now, we can apply (ii) to m and the set $\{y_1, \ldots, y_n\}$. □

In general, one can characterize medians adapting a result from Ref. 3; see Theorem 3.1 in Ref. 3.

Proposition 3.3. A point $m \in X, m \notin A$, is a general median of A if and only if there exist n norm-one functionals $\|u_i\|_i^o = 1$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} u_i = 0$ and $u_i(m - a_i) = \|m - a_i\|_i$, $i = 1, \ldots, n$.

Moreover, we can derive also a useful condition for the coincidence of a general median with one of the points in A.

Theorem 3.1. The point $a_1 \in A$ is a general median of A if and only if there exist $n-1$ norm-one functionals u_i $i=2,\dots,n$, such that $u_i(a_1 - a_i) = \|a_1 - a_i\|_i$ and $\|\sum_{i=2}^n u_i\|_1^o \le 1$.

Proof. From the first-order optimality condition, a_1 is a general median iff there exist n functionals $u_i \in \partial\|a_1 - a_i\|_i$ such that $\sum_{i=1}^n u_i = 0$. The condition

$$u_i \in \partial\|a_1 - a_i\|_i, \quad \text{for } i = 2,\dots,n,$$

is equivalent to

$$u_i(a_1 - a_i) = \|a_1 - a_i\|_i, \quad \|u_i\|_i^o = 1,$$

while for $i = 1$ it means that

$$u_1 \in B_1^o := \{u : \|u\|_1^o \le 1\}.$$

Now, take

$$u_1 = -\sum_{i=2}^n u_i;$$

then,

$$\|\sum_{i=2}^n u_i\|_1^o \le 1.$$

Conversely, assume that there exist $u_i \in \partial\|a_1 - a_i\|_i$ for all $i = 2,\dots,n$ such that $\|\sum_{i=2}^n u_i\|_1^o \le 1$. Thus, there exists $u_1 \in B_1^o$ such that

$$u_1 = -\sum_{i=2}^n u_i,$$

i.e.,

$$\sum_{i=1}^n u_i = 0.$$

Hence, a_1 is a general median of A. □

Corollary 3.2. (Majority Theorem.) Assume that $\|\cdot\|_i = w_i \|\cdot\|_1$, $w_i \geq 0$ for all $i = 1, \ldots, n$. If there exists a_j such that $w_j > \sum_{i \neq j} w_i$, then a_j is a general median of A.

It is known that, when a single norm is used and

$$A = \{a_1, a_2\},$$

then

$$M(A) = [a_1, a_2]_{\|\cdot\|} := \{x : \|a_1 - x\| + \|x - a_2\| = \|a_1 - a_2\|\}.$$

The set $[a_1, a_2]_{\|\cdot\|}$ is usually called the metric segment of the norm between a_1 and a_2 (see Ref. 23). Obviously, this result cannot be extended to deal with general medians, because here each point uses a different norm. Nevertheless, there exists a way to generalize the concept of metric segment using tools from convex analysis. Let

$$N_i(p) = \{x : \langle p - u, x \rangle \leq 0, \ \ \forall u \in B_i^o\},$$

the normal cone to the dual ball of B_i at p.

Proposition 3.4. The metric segment

$$[a, b]_{\|\cdot\|} = \{x : \|a_1 - x\| + \|x - a_2\| = \|a_1 - a_2\|\}$$

coincides with the set $(a_1 + N(p_1)) \cap (a_2 + N(p_2))$, with $p_i \in B_i^o$, $i = 1, 2$, and $p_1 + p_2 = 0$.

Proof. Since

$$\inf_{x \in X} (\|x - a_1\| + \|x - a_2\|) \geq \|a_1 - a_2\|,$$

then $x \in [a_1, a_2]_{\|\cdot\|}$ iff there exist $p_i \in \partial \|x - a_i\|$ for $i = 1, 2$ such that $p_1 + p_2 = 0$ (apply $[a_1, a_2]$ the first-order optimality condition). Now,

$$p_i \in \partial \|x - a_i\|, \text{ for } i = 1, 2, \text{ iff } p_i \in B_i^o \text{ and } p_i(x - a_i) = \|x - a_i\|, i = 1, 2.$$

Since

$$\|x\| = \sup_{u \in B^o} u(x),$$

it holds that

$$\{x : p_i(x - a_i) = \|x - a_i\|\} = a_i + N(p_i), \quad \text{for } i = 1, 2.$$

This proves the result. □

Therefore, given a_1, a_2 with their respective norms $\|\cdot\|_i, i = 1, 2$, the metric segment, which depends now on the respective norms at a_1, a_2 is

$$[(a_1, \|\cdot\|_1); (a_2, \|\cdot\|_2)] = (a_1 + N_1(p_1)) \cap (a_2 + N_2(p_2)),$$

$$\text{for some } p_i \in B_i^o, \quad i = 1, 2, \quad p_1 + p_2 = 0.$$

First of all, we note in passing that the above concept is well-defined; i.e., the metric segment is uniquely defined. Indeed, by Proposition 3.3 $[(a_1, \|\cdot\|_1); (a_2, \|\cdot\|_2)] \neq \emptyset$ coincides with the set of minimizers of the problem

$$\min_{x \in X}(\|x - a_1\|_1 + \|x - a_2\|_2);$$

this is a consequence of Theorem 3.1 in Ref. 3. Therefore, our definition is correct, since the set of minimizers is well-defined regardless of the norm-one functionals used in its description.

These metric segments contain always the rectilinear segment when a single norm is used. This establishes a new difference with respect to the case considered in this paper. When different norms are used, the metric segments may reduce to a singleton as it is shown in the following example.

Example 3.3. Consider on the plane

$$a_1 = (0, 0), \quad a_2 = (1, 0),$$

with

$$\|x\|_1 = 2\|x\|_2, \quad \text{for any } x \in \mathbb{R}^2.$$

In this case, it is clear that

$$[(a_1, \|\cdot\|_1), (a_2, \|\cdot\|_2)] = \{a_1\}.$$

Also, in the single-norm case in the plane, it is well-known that, if $A = \{a_1, a_2, a_3, a_4\}$ forms a convex quadrilateral, then the intersection of the two diagonals $[a_1, a_3] \cap [a_2, a_4]$ is a minisum point of A, a point minimizing the sum of the distances from A; see Chapter 3 in Ref. 24. This results fails to be true with different norms.

Example 3.4. Let

$$A = \{a_1, a_2, a_3, a_4\},$$

with

$$a_1 = (0, 0), \quad a_2 = (1, 0), \quad a_3 = (1, 1), \quad a_4 = (0, 1),$$

and norms associated with the different points given by

$$\|x\|_i = (1/2^i) l_2(x), \qquad \text{for all } i = 1, \dots, 4,$$

$$l_2(x) = \sqrt{x_1^2 + x_2^2} \text{ is the Euclidean norm in the plane.}$$

With these norms, the minisum problem is strictly convex. Thus, there exists a unique median point of A. Then, Corollary 3.2 ensures that the unique median is the point a_1.

We can give also some positive results regarding metric segments. The following result extends a characterization of median points given in Ref. 23 for the single norm case. Notice that the proof is new and simpler than the one given in Ref. 23.

Theorem 3.2. If A can be partitioned into a set of pairs $\{a_i, a_j\}$ such that

$$\bigcap_{\{a_i, a_j\} \subset A, a_i \neq a_j} [(a_i, \|\cdot\|_i); (a_j, \|\cdot\|_j)] \neq \emptyset,$$

then this intersection is exactly the median set of A.

Proof. The set of medians of A is given by the minimizers of the following problem:

$$\inf_{x \in X} \sum_{i=1}^{n} \|x - a_i\|_i.$$

By hypothesis, there exists a partition of A such that

$$\bigcap_{\{a_i, a_j\} \subset A, a_i \neq a_j} [(a_i, \|\cdot\|_i); (a_j, \|\cdot\|_j)] \neq \emptyset.$$

By the arguments above,

$$[(a_i, \|\cdot\|_i); (a_j, \|\cdot\|_j)] = \arg\min_{x\in X}(\|x-a_i\|_i + \|x-a_j\|_j),$$

for any pair $\{a_i, a_j\}$ of the partition of A. Then, it is clear that

$$\arg\min_{x\in X}\sum_{i=1}^{n}\|x-a_i\|_i = \bigcap_{\{a_i,a_j\}\in \text{Partition}} [(a_i, \|\cdot\|_i); (a_j, \|\cdot\|_j)]. \qquad \square$$

Proposition 3.5. Let $\|\cdot\|_i$ be Hilbertian norms, $i=1,\dots,n$. Let m be a median of A and $\|m-a_i\|_i = k$, for $i=1,\dots,n$ and some $k\in\mathbb{R}^+$. Then, we have

$$\sum_{i=1}^{n}\|m\|_i^2 = \sum_{i=1}^{n}\|a_i\|_i^2 - nk^2.$$

Proof. Let m be a median of A. According to Proposition 3.3, there exist functionals $u_i, i=1,\dots n$ satisfying

$$\sum_{i=1}^{n} u_i = 0 \quad \text{and} \quad u_i(y) = (m-a_i, y)_i/k,$$

where $(\cdot,\cdot)_i$ denotes the scalar product defining $\|.\|_i$. Therefore,

$$\sum_{i=1}^{n} u_i = 0$$

means that

$$\sum_{i=1}^{n}(m, y)_i = \sum_{i=1}^{n}(a_i, y)_i, \quad \text{for every } y\in X.$$

By taking $y=m$, we obtain

$$\begin{aligned}2\sum_{i=1}^{n}\|m\|_i^2 &= 2\sum_{i=1}^{n}(a_i, m)_i \\ &= -\left(\sum_{i=1}^{n}\|a_i-m\|_i^2 - \sum_{i=1}^{n}\|a_i\|_i^2 - \sum_{i=1}^{n}\|m\|_i^2\right) \\ &= -nk^2 + \sum_{i=1}^{n}\|a_i\|_i^2 + \sum_{i=1}^{n}\|m\|_i^2,\end{aligned}$$

which is the thesis. $\square$

Finally, we conclude the paper showing some relationships between general medians and centers. Set

$$\begin{aligned} s(A) &= \sum_{1\le i,j\le n} \|a_i - a_j\|_i \\ &= \sum_{1\le i<j\le n} (\|a_i - a_j\|_i + \|a_i - a_j\|_j), \\ s_j(A) &= \sum_{i=1}^{n} \|a_j - a_i\|_i \\ &= n\mu(A, a_j), \text{ for } j = 1, \dots, n. \end{aligned}$$

Therefore,

$$s(A) = \sum_{j=1}^{n} s_j(A).$$

Moreover,

$$\begin{aligned} s(A) &= n \sum_{i=1}^{n} \mu(A, a_i) \\ &\le n \left(((n-1)/n) \sum_{i=1}^{n} r_i(A) \right) \\ &\le (n-1) \sum_{i=1}^{n} \delta_i(A) \\ &\le (n-1) n \delta(A), \end{aligned}$$

so

$$\begin{aligned} \mu(A) &\le \frac{\sum_{i=1}^{n} \mu(A, a_i)}{n} \\ &\le [(n-1)/n]\,\delta(A). \end{aligned}$$

In addition, we get the following result that implies the same relationship.

Proposition 3.6. Given $A = \{a_1, \dots, a_n\}$, let $g = (1/n)\sum_{i=1}^{n} a_i$, where g is the centroid of A. Then, we have

$$\begin{aligned} \mu(A, g) &\le (1/n) \sum_{i=1}^{n} \mu(A, a_i) \\ &= s(A)/n^2. \end{aligned}$$

Proof. The proof follows from the convexity of $\mu(A,x)$. □

Finally, the inequality

$$\mu(A) \leq [(n-1)/n]\,\delta(A), \quad \#A = n,$$

follows from Proposition 3.6.

References

1. Carrizosa, E., and Puerto, J., *A Discretizing Algorithm for Location Problems*, European Journal of Operational Research, Vol. 80, pp. 166–174, 1995.
2. Durier, R., *The General One-Center Location Problem*, Mathematics of Operations Research, Vol. 20, pp. 400–414, 1995.
3. Durier, R., and Michelot, C., *Geometrical Properties of the Fermat–Weber Problem*, European Journal of Operational Research, Vol. 20, pp. 332–343, 1985.
4. Durier, R., *On Pareto Optima, the Fermat-Weber Problem, and Polyhedral Gauges*, Mathematical Programming, Vol. 47, pp. 65–79, 1990.
5. Durier, R., *A General Framework for the One-Center Location Problem*, Lecture Notes in Economics and Mathematical Systems, Springer, New York, NY, Vol. 382, pp. 441–457, 1992.
6. Nickel, S., *Discretization of Planar Location Problems*, Verlag Shaker, Aachen, Germany, 1995.
7. Frenk, J. B. G., Gromicho, J., and Zhang, S., *General Models in Min-Max Continuous Location: Theory and Solution Techniques*, Journal of Optimization Theory and Applications, Vol. 89, pp. 39–63, 1996.
8. Frenk, J. B. G., Gromicho, J., and Zhang, S., *General Models in Min-Max Planar Location: Checking Optimality Conditions*, Journal of Optimization Theory and Applications, Vol. 89, pp. 65–87, 1996.
9. Gromicho, J., *Quasiconvex Optimization and Location Theory*, Applied Optimization, Kluwer Academic Publishers, Dordrecht, Holland, Vol. 9, 1998.
10. Parlar, M., *Single-Facility Location Problems with Region-Dependent Distance Metrics*, International Journal of System Sciences, Vol. 25, pp. 513–525, 1994.
11. Drezner, Z., and Wesolowski, G. O., *Sensitivity Analysis to the Value of p of the l_p Distance Weber Problem*, Annals of Operations Research, Vol. 111, pp. 135–150, 2002.
12. Wendell, R. E., and Hurter, A. P., Jr., *Location Theory, Dominance, and Convexity*, Operations Research, Vol. 21, 314–320, 1973.
13. Hansen, P., Perreur, J., and Thisse, J. F., *Location Theory, Dominance, and Convexity: Some Further Results*, Operations Research, Vol. 28, pp. 1241–1250, 1980.
14. Carrizosa, E., and Fernàndez, F. R., *A Polygonal Upper Bound for the Efficient Set for Single-Facility Location Problems with Mixed Norms*, TOP, Vol. 1, pp. 107–116, 1993.

15. Rodríguez-Chía, A. M., and Puerto, J., *Geometrical Description of the Weakly Efficient Solution Set for the Multicriteria Location Problem*, Annals of Operations Research, Vol. 111, pp. 181–196, 2002.
16. Papini, P. L., *Existence of Centers and Medians*, 2004 TOP, 2004 (to appear).
17. Abadie, J., Editor, *Nonlinear Programming*, North-Holland, Amsterdam, Holland, 1967.
18. Gritzman, P., and Klee, V., *Mathematical Programming and Convex Geometry*, Handbook of Convex Geometry, Vol. A, Edited by P. M. Gruber and J. M. Wills, North-Holland, Amsterdam, Holland, 1993.
19. Domínguez-Benavides, T., *Normal Structure Coefficients of* $L^p(\Omega)$, Proceedings of the Royal Society of Edimburgh, Vol. 117 A, pp. 299–303, 1991.
20. Papini, P. L., *How Can We Locate Centers*?, Studies on Locational Analysis, Vol. 7, pp. 143–152, 1994.
21. Baronti, M., and Papini, P. L., *Remotal Sets Revisited*, Taiwanese Journal of Mathematics, Vol. 5, pp. 367–373, 2001.
22. Baronti, M., Casini, E., and Papini, P. L., *Central Points of Finite Sets*, Rendiconti del Circolo Matematico di Palermo (II), Supplemento, Vol. 40, pp. 63–72, 1996.
23. Martini, H., Swanepoel, K. J., and Weiss, G., *The Fermat-Torricelli Problem in Normed Planes and Spaces*, Journal of Optimization Theory and Applications, Vol. 115, pp. 283–314, 2002.
24. Cieslik, D., *Steiner Minimal Trees*, Nonconvex Optimization and Its Applications, Kluwer, Dordrecht, Holland, Vol. 23, 1998.